

THE ADJOINT VARIETY OF $\mathrm{SL}_{m+1}\mathbb{C}$ IS RIGID TO ORDER THREE

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ABSTRACT. I prove that the adjoint variety of $\mathrm{SL}_{m+1}\mathbb{C}$ in $\mathbb{P}(\mathfrak{sl}_{m+1}\mathbb{C})$ is rigid to order three.

The principle result of this paper is Theorem 4.6 (page 8) which asserts that the adjoint variety of the simple Lie group $\mathrm{SL}_{m+1}\mathbb{C}$ is rigid to order three. The result is extrinsic; roughly speaking, if a variety $Y \subset \mathbb{P}(\mathfrak{sl}_{m+1}\mathbb{C}) = \mathbb{P}^{m^2+2m-1}$, of dimension $n = 2m - 1$, resembles the adjoint variety to third order at a 3-general point $y \in Y$, then there is a transformation in $\mathrm{GL}_{m^2+2m}\mathbb{C}$ mapping Y onto the adjoint variety.

The conclusion is significant because it is the first rigidity result for a variety with non-vanishing Fubini cubic F_3 (a third order invariant). And it is striking that this is the first example of k -th order rigidity for which the $(k + 1)$ -th order Fubini invariant is nonzero: F_4 can not be normalized to zero.

The proof is based on the E. Cartan's method of moving frames. The reader may find similar applications of the technique to the study of submanifolds of \mathbb{CP}^N in [4, 9, 10, 11, 12], and their references. The paper is organized as follows

- §1 Notation is set. The first-order adapted frame bundle associated to a variety is introduced, and the relative differential invariants F_k , or Fubini forms, are discussed. (These invariants describe the lines osculating to a variety with order k .)
- §2 The notions of *agreement to k -th order* and *rigidity* are made explicit, and previous rigidity results are reviewed.
- §3 The frame bundle normalizations necessary to our computations are sketched.
- §4 The adjoint variety of $\mathrm{SL}_{m+1}\mathbb{C}$ is introduced and the main result, Theorem 4.6, stated.
- §5 Theorem 4.6 is proven.
- §6 An open question: what rigidity results might we expect for the other adjoint varieties?

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1. PRELIMINARIES

1.1. Notation. The aim of this section is merely to establish notation; see [7, §3] for a thorough discussion of the ideas presented here. I will, for the most part, follow the notation and conventions of [7, 9].

Let V be a complex vector space of dimension $N + 1$, and $X^n \subset \mathbb{P}V$ a variety of (complex) dimension n . Fix the index ranges

$$\begin{aligned} 0 &\leq I, J, K, \dots \leq N \\ 1 &\leq a, b, c, \dots \leq n \\ n + 1 &\leq u, v, w \dots \leq N \end{aligned}$$

The Einstein convention holds here: repeated indices, one raised and one lowered, are summed over. Given a collection of vectors $\{v_j\}$ in a vector space, let

$$\langle v_j \rangle = \text{span}\{v_j\}$$

denote their span. Given a nonzero $v \in V$, I will denote by $[v]$ its projection to $\mathbb{P}V$. The *cone over* $X \subset \mathbb{P}V$ is

$$\widehat{X} = \{v \in V \setminus \{0\} : [v] \in X\}.$$

1.2. A frame bundle over X . Let $X^n \subset \mathbb{P}V$ be a variety. The bundle of *first-order adapted frames* \mathcal{F}^1 over

$$M := X_{\text{smooth}} \subset X$$

is the subset of those $\mathbf{e} = (e_0, e_1, \dots, e_N) \in GL(V)$ for which $e_0 \in \widehat{M}$, and the *affine tangent space* to the cone over M , $T_{e_0}\widehat{M} \subset V$, is spanned by $\{e_0, e_1, \dots, e_n\}$. Let ω denote the pull-back of the Maurer-Cartan form on $GL(V)$ to \mathcal{F}^1 . In particular,

$$de_I = \omega_I^J e_J \quad \text{and} \quad d\omega_K^I = -\omega_J^I \wedge \omega_K^J.$$

Since the $T_{e_0}\widehat{M} = \langle e_0, e_a \rangle$, we must have

$$(1.1) \quad \omega_0^u = 0;$$

and the forms $\omega_0^0, \omega_0^1, \dots, \omega_0^n$ are linearly independent. Differentiating this formula and an application of Cartan's Lemma (cf. [1] or [7, Lem. A.1.9]) produces

$$(1.2) \quad \omega_a^u = q_{ab}^u \omega_0^a,$$

where, $q_{ab}^u = q_{ba}^u$ are \mathbb{C} -valued functions on \mathcal{F}^1 . The q_{ab}^u are the coefficients of the *second fundamental form*

$$II = q_{ab}^u \omega_0^a \omega_0^b \otimes \underline{e}_u \in \Gamma(\mathcal{F}^1, \pi^*(S^2 T^*M \otimes NM)),$$

a section of the pulled-back bundle $\pi^*(S^2 T^*M \otimes NM)$ over \mathcal{F}^1 . Here TM and NM denote the tangent and normal bundles over M . Given $x = [e_0] \in M$, $T_x M \simeq \widehat{x}^* \otimes (T_{e_0}\widehat{M}/\widehat{x})$; the normal space $N_x M = T_x \mathbb{P}V / T_x M$ is spanned by $\underline{e}_u := e^0 \otimes (e_u \bmod T_{e_0}\widehat{M})$; and the $e^I \in V^*$ are dual to the $e_I \in V$. A priori defined on \mathcal{F}^1 , the second fundamental form descends to a well-defined section of $S^2(T^*M) \otimes NM$ over M .

Similarly, differentiating (1.2) yields the *Fubini cubic form*

$$F_3 = r_{abc}^u \omega_0^a \omega_0^b \omega_0^c \otimes \underline{e}_u \in \Gamma(\mathcal{F}^1, \pi^*(S^3 T^*M \otimes NM)),$$

a higher order differential invariant. The coefficients $r_{abc}^u : \mathcal{F}^1 \rightarrow \mathbb{C}$ of F_3 are fully symmetric in the lower indices, and are defined by

$$(1.3) \quad r_{abc}^u \omega_0^c = -dq_{ab}^u - q_{ab}^u \omega_0^0 - q_{ab}^v \omega_v^u + q_{ae}^u \omega_b^e + q_{be}^u \omega_a^e.$$

Continuing in this fashion we may construct a sequence of higher order differential invariants

$$F_k = r_{a_1 \dots a_k}^u \omega_0^{a_1} \dots \omega_0^{a_k} \otimes \underline{e}_u \in \Gamma(\mathcal{F}^1, \pi^*(S^k T^*M \otimes NM)), \quad k \geq 3.$$

Proposition 1.4. *Set $p = 2q$ if p is even, and $p = 2q + 1$ if p is odd. Then the coefficients of F_{p+1} , again fully symmetric in the lower indices, are defined by*

$$\begin{aligned} r_{a_1 \dots a_p b}^u \omega_0^b &= -dr_{a_1 \dots a_p}^u - (p-1)r_{a_1 \dots a_p}^u \omega_0^0 - r_{a_1 \dots a_p}^v \omega_v^u \\ &\quad + \mathfrak{S}_{a_1 \dots a_p} \left\{ (p-2)r_{a_1 \dots a_{p-1}}^u \omega_{a_p}^0 + r_{a_1 \dots a_{p-1}b}^u \omega_{a_p}^b - q_{a_p b}^u r_{a_1 \dots a_{p-1}}^v \omega_v^b \right\} \\ &\quad - \sum_{k=q+1}^{p-2} \mathcal{S}_{a_1 \dots a_k, a_{k+1} \dots a_p} \left\{ \left(r_{a_1 \dots a_k b}^u r_{a_{k+1} \dots a_p}^v + r_{a_{k+1} \dots a_p b}^u r_{a_1 \dots a_k}^v \right) \omega_v^b \right. \\ &\quad \left. + \left((k-1)r_{a_1 \dots a_k}^u r_{a_{k+1} \dots a_p}^v + (p-k-1)r_{a_{k+1} \dots a_p}^u r_{a_1 \dots a_k}^v \right) \omega_v^0 \right\} \\ &\quad - \binom{p+1}{\text{mod } 2} \mathcal{S}_{a_1 \dots a_q, a_{q+1} \dots a_p} \left\{ r_{a_1 \dots a_q b}^u r_{a_{q+1} \dots a_p}^v \omega_v^b + (q-1)r_{a_1 \dots a_q}^u r_{a_{q+1} \dots a_p}^v \omega_v^0 \right\}. \end{aligned}$$

As in [7] \mathfrak{S} denotes cyclic summation over the indices. The notation \mathcal{S} represents a symmetrizing operation defined as follows: Given symmetric tensors $T_{a_1 \dots a_k}$ and $U_{b_1 \dots b_\ell}$, $\mathcal{S}_{a_1 \dots a_k, b_1 \dots b_\ell}$ denotes summation over the $\binom{k+\ell}{k} = \binom{k+\ell}{\ell}$ elements of the symmetric group on $k+\ell$ elements symmetrizing the product. For example, $\mathcal{S}_{a_1 \dots a_k, b} = \mathfrak{S}_{a_1 \dots a_k b}$ and $\mathcal{S}_{ab, cd} = \mathfrak{S}_{abc} \cdot \mathfrak{S}_{bcd}$.

Notice that the last line in the equation appears only when p is even. Also, I've used the convention $II = F_2$ and $q_{ab}^u = r_{ab}^u$.

Remark. This formula corrects errors in the expression for $r_{a_1 \dots a_p b}^u \omega_0^b$ given in [7, p. 108] and [9, (2.20)].

Proof. The formula is easily verified for F_3 and F_4 . (For F_3 it is necessary to use the convention that $r_a^u = 0$.) Obtain the coefficients of F_4

$$(1.5) \quad r_{abcd}^u \omega_0^d = -dr_{abc}^u - 2r_{abc}^u \omega_0^0 - r_{abc}^v \omega_v^u + \mathfrak{S}_{abc} (r_{abe}^u \omega_c^e + q_{ab}^u \omega_c^0 - q_{ae}^u q_{bc}^v \omega_v^e)$$

by differentiating (1.3) and another application of Cartan's Lemma. The general statement is established by induction; once for p even, and once for p odd. \square

The formula for F_k given by Proposition 1.4 provides a slight improvement of [9, Prop. 2.41]:

Corollary 1.6. *Let x be a smooth point of a variety $X \subset \mathbb{P}V$. Suppose that there exists a framing $e_x = (e_0, \dots, e_N)$ over $x = [e_0]$ where the coefficients of $F_\ell, F_{\ell+1}, \dots, F_{2\ell-1}$ all vanish. If x is a $(2\ell-1)$ -general point, then the coefficients of F_k vanish on $\mathcal{F}^{2\ell-1}$, for all $k \geq \ell$.*

Remark. Loosely speaking, after taking k derivatives there will be both discrete and continuous invariants: the discrete invariants are locally constant at a k -general point.

Proof. This follows immediately from Proposition 1.4. \square

Unlike the second fundamental form, the $F_k \in \Gamma(\mathcal{F}^1, \pi^*(S^k T^*X \otimes NX))$, $k \geq 3$, do not descend to well-defined sections over X . For this reason we call them *relative differential invariants*. However:

- Let $|F_{x, \ell}| = F_\ell(N_x X) \subset S^\ell(T_x^*X)$. The zero locus $\mathcal{C}_{k, x}$ of $|II_x|, |F_{3, x}|, \dots, |F_{k, x}|$ in $\mathbb{P}(T_x X)$ is well defined, and consists of the tangent directions to lines making contact to order k with X at x . See §2 below.
- By restricting $F_3 : N^*X \rightarrow S^3 T^*X$ to the kernel of $F_2 = II : N^*X \rightarrow S^2 T^*X$ we obtain a tensor $\mathbb{F}_3 = III \in S^3 T^*X \otimes N_3$ on X . Here $N_3 = T_x \mathbb{P}V / \{T_x X \oplus II(S^2 T_x X)\}$, and \mathbb{F}_3 is called the *third fundamental form*. A series of higher order fundamental forms $\mathbb{F}_k \in S^k T^*X \otimes N_k$ on X is defined inductively. See [7, §3.5] for details.

2. GRIFFITHS–HARRIS RIGIDITY

The F_k play an important role in establishing the rigidity of a variety.

Proposition 2.1 ([7, Cor. 3.7.2]). *A complex projective variety $X \subset \mathbb{P}V$ is uniquely determined up to projective equivalence by the infinite sequence of relative differential invariants at a smooth point $x \in X$.*

The proof of the proposition is straightforward. Here is a sketch. Near a smooth point x , X may be expressed locally as a graph $x^u = f^u(x^1, \dots, x^n)$. There exists a local section $s : X \rightarrow \mathcal{F}^1$ so that the pull-back of F_k is $s^*(F_k) = (-1)^k \frac{\partial^k f^u}{\partial x^{a_1} \dots \partial x^{a_k}} dx^{a_1} \dots dx^{a_k} \otimes \frac{\partial}{\partial x^u}$ at x . In particular, the coefficients of F_k determine the Taylor series of the f^u , which in turn determine X .

Definition 2.2. *Two varieties $X^n, Y^n \subset \mathbb{P}^N$ agree to order k at $x \in X_{\text{smooth}}$ and $y \in Y_{\text{smooth}}$ if there exist frames $\mathbf{e}_x \in \mathcal{F}_X^1$ and $\mathbf{e}_y \in \mathcal{F}_Y^1$ over x and y such that the coefficients of $F_\ell(\mathbf{e}_x)$ and $F_\ell(\mathbf{e}_y)$ are equal for all $\ell \leq k$.*

Notice this implies $\mathcal{C}_{\ell,x} = \mathcal{C}_{\ell,y}$, $\ell \leq k$.

Definition 2.3. *When agreement to k -th order forces agreement to all orders, then Proposition 2.1 implies that X and Y are projectively equivalent, and we say X is rigid to order k .*

Remark. Any variety X meeting the conditions of Corollary 1.6 is rigid to order $k = 2\ell - 1$.

Here is another perspective on rigidity. All of the varieties X to be discussed in this paper admit sub-bundles $\mathcal{F}_X \subset \mathcal{F}_X^1$ of the first-order adapted frames on which the coefficients of the F_k are constant. The sub-bundles \mathcal{F}_X are maximal integral submanifolds of a Frobenius system \mathcal{I} on $\text{GL}(V)$. The Frobenius system is generated by constant coefficient linear combinations of the entries of the Maurer-Cartan form on $\text{GL}(V)$. In particular, any other maximal integral submanifold of \mathcal{I} is of the form $g \cdot \mathcal{F}_X$, $g \in \text{GL}(V)$.

Suppose Y^n agrees with X^n (both algebraic varieties of \mathbb{P}^N) to order k at a k -general point $y \in Y_{\text{smooth}}$. Equivalently, we may restrict to a sub-bundle $\mathcal{F}_Y^k \subset \mathcal{F}_Y^1$ over an open neighborhood $U \subset Y_{\text{smooth}}$ of y on which the coefficients of $F_{\ell,Y}$ are equal to the constant coefficients of $F_{\ell,X}$, $\ell \leq k$. The variety X will be rigid to order k if and only if we may further reduce (or *normalize*, see §3) \mathcal{F}_Y^k to a sub-bundle \mathcal{F}_U which is an integral submanifold of \mathcal{I} . When this is the case, we have $g \cdot \mathcal{F}_U \subset \mathcal{F}_X$ yielding a projective linear transformation $g \cdot U \subset X$. As we are working with algebraic varieties, it now follows that $g \cdot Y = X$.

2.1. Previous rigidity results.

2.1.1. The Segre variety. Let W_1^* and W_2 be complex vector spaces of dimensions $d_1, d_2 > 1$, respectively. The Segre variety $X = \text{Seg}(\mathbb{P}W_1^* \times \mathbb{P}W_2) = \{[w_1^* \otimes w_2]\} \subset \mathbb{P}(W_1 \otimes W_2)$ is the set $\mathbb{P}\{\text{rank one linear maps } W_1 \rightarrow W_2\}$. Given $d_2 > 2$, Landsberg has shown that X is rigid to order 2 [10]. For all $k \geq 3$, the F_k may be normalized to zero; and $\mathcal{C}_x = \mathcal{C}_{2,x} = \mathbb{P}^{d_1-1} \sqcup \mathbb{P}^{d_2-1}$ is the disjoint union of two linear subspaces.

2.1.2. The Veronese variety. Let W be a complex vector space of dimension $n+1 > 2$, and consider the Veronese embedding $X = v_2(\mathbb{P}W) \subset \mathbb{P}(S^2 W)$ of $\mathbb{P}W$. In this case $|II_x| = S^2 T_x^* X$, so that $\mathcal{C}_{2,x} = \emptyset$, and there exists a sub-bundle of the first order adapted frames upon which the $F_k = 0$, $k \geq 3$. Landsberg has shown that $v_2(\mathbb{P}W)$ is rigid to order three [10]. (The case $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ was established by Griffiths and Harris [4].) Note that, in the case that $n = 2m - 1$, X is the adjoint variety of the simple Lie group $\text{Sp}(W)$.

2.1.3. Compact Hermitian symmetric spaces. It is a classic result of Fubini that a quadric hypersurface (of rank > 1) in \mathbb{P}^N , $N > 2$, is rigid to order three; cf. [3] or [7, Th. 3.9.1]. (In contrast, Monge showed that the conics in \mathbb{P}^2 are rigid to order five [9, §3.6].) The quadric hypersurfaces are rank two compact Hermitian symmetric spaces. Landsberg has shown that any other rank two CHSS, in its minimal homogeneous embedding, is rigid to order two [11].

Shortly after Landsberg's preprint appeared Hwang and Yamaguchi proved the following theorem: Let $Y^n \subset \mathbb{CP}^N$ be a complex submanifold and $y \in Y$ a general point. Suppose that X is an irreducible rank r compact Hermitian symmetric space in its natural embedding, other than a quadric hypersurface. If there exist linear maps $f : T_x X \rightarrow T_y Y$ and $g_k : N_{k,x} X \rightarrow N_{k,y} Y$ such that the induced maps $S^k T_x^* X \otimes N_{k,x} X \rightarrow S^k T_y^* Y \otimes N_{k,y} Y$ take $\mathbb{F}_{k,x} \rightarrow \mathbb{F}_{k,y}$, for $2 \leq k \leq r$, then \bar{Y} is projectively equivalent to X [6]. Unlike the other results mentioned in this section, Hwang and Yamaguchi's proof is not based on a moving frame calculation. Their result follows from an elegant application of representation theory, based on a theory developed by Se-ashi [14], and a Lie algebra cohomology computation made easy by the work of Kostant [8].

The fundamental forms \mathbb{F}_k are the only nonzero invariants of the CHSS. In particular, the F_k , $k \geq 3$, may be normalized to zero, so that $\mathcal{C}_x = \mathcal{C}_{2,x}$.

The interested reader will find *intrinsic* rigidity results for the compact Hermitian symmetric spaces in the paper [5] of Hwang and Mok.

3. NORMALIZATIONS

Given a frame \mathbf{e} in the fibre over a smooth point $x = [e_0] \in X$, consider the fibre motion $\tilde{\mathbf{e}} = g \cdot \mathbf{e}$, $g \in GL(V)$. It will be helpful to compare the expressions for F_k at \mathbf{e} and $\tilde{\mathbf{e}}$. Transformations by block diagonal matrices $\tilde{e}_0 = g_0^0 e_0$, $\tilde{e}_a = g_a^b e^b$, and $\tilde{e}_u = g_u^v e_v$ do not change the F_k . For example, if the coefficients of II are q_{ab}^u at e , then the coefficients of II at \tilde{e} are $\tilde{q}_{ab}^u = h_0^0 h_v^u g_a^c g_b^d q_{cd}^v$, and $II_e = II_{\tilde{e}}$. So we consider transformations of the form

$$(3.1) \quad \tilde{e}_0 = e_0, \quad \tilde{e}_a = e_a + g_a^0 e_0, \quad \tilde{e}_u = e_u + g_u^0 e_0 + g_u^a e_a.$$

Let $\tilde{\omega}$ denote the Maurer-Cartan forms at $\tilde{e} \in \mathcal{F}^1$, so that $d\tilde{e} = \tilde{\omega}\tilde{e}$. Making use of $\tilde{e} = g e$, we derive $\tilde{\omega} = g^{-1}\omega g + g^{-1}dg$. Explicitly,

$$(3.2) \quad \begin{aligned} \tilde{\omega}_0^a &= \omega_0^a & \tilde{\omega}_b^u &= \omega_b^u \\ \tilde{\omega}_0^0 &= \omega_0^0 - g_c^0 \omega_0^c & \tilde{\omega}_b^a &= \omega_b^a + g_b^0 \omega_0^a - g_t^a q_{bc}^t \omega_0^c \\ \tilde{\omega}_v^u &= \omega_v^u + g_v^e q_{ce}^u \omega_0^c & \tilde{\omega}_v^a &= \omega_v^a + (\delta_v^a g_v^0 - g_v^e g_t^a q_{ce}^t) \omega_0^c + g_v^c \omega_c^a - g_t^a \omega_v^t + dg_v^a \\ \tilde{\omega}_b^0 &= \omega_b^0 + (q_{bc}^t (g_e^0 g_t^e - g_t^0) - g_b^0 g_c^0) \omega_0^c + g_b^0 \omega_0^0 - g_c^0 \omega_b^c + dg_b^0 \\ \tilde{\omega}_v^0 &= \omega_v^0 + (g_v^e (g_d^0 g_t^d - g_t^0) q_{ce}^t - g_v^0 g_c^0) \omega_0^c + g_v^0 \omega_0^0 + g_v^c \omega_c^0 - g_c^0 (g_v^e \omega_e^c + \omega_v^c) \\ &\quad + (g_e^0 g_t^e - g_t^0) \omega_v^t + dg_v^0 - g_e^0 dg_v^e \end{aligned}$$

Using the two expressions on the first line we immediately see that

$$(3.3) \quad \tilde{q}_{ab}^u = q_{ab}^u.$$

The coefficients of F_3 at $\tilde{e} \in \mathcal{F}^1$ are given by (1.3)

$$\tilde{r}_{abc}^u \tilde{\omega}_0^c = -d\tilde{q}_{ab}^u - \tilde{q}_{ab}^u \tilde{\omega}_0^0 - \tilde{q}_{ab}^v \tilde{\omega}_v^u + \tilde{q}_{ae}^u \tilde{\omega}_b^e + \tilde{q}_{be}^u \tilde{\omega}_a^e.$$

Replace the \tilde{q} and $\tilde{\omega}$ with their q and ω expressions. After simplifying, both sides of the equation are seen to be linear combinations of the ω_0^c . Equating coefficients produces

$$(3.4) \quad \tilde{r}_{abc}^u = r_{abc}^u + \mathfrak{S}_{abc}(\delta_v^u g_a^0 - q_{ae}^u g_v^e) q_{bc}^v.$$

Consequently, we see that although the subspace $|F_3| \subset S^3 T_x^* X$ is not well-defined over x (the subspace moves as we vary the frame e over x), it is well-defined over x modulo $|II| \circ T^* X$. In particular, the zero locus $\mathcal{C}_{3,x} \subset \mathbb{P}(T_x X)$ of $\{|II|, |F_3|\}$ is well-defined.

Similarly (1.5,3.2) yield,

$$\begin{aligned} \tilde{r}_{abcd}^u &= r_{abcd}^u + \mathfrak{S}_{abcd} \left\{ 2 r_{abc}^u g_d^0 - r_{abc}^v q_{de}^u g_v^e \right\} \\ &\quad + \mathcal{S}_{a,b,cd} \left\{ (\delta_v^u g_a^0 - q_{ae}^u g_v^e) (\delta_w^v g_b^0 - q_{be}^v g_w^e) q_{cd}^w \right\} \\ &\quad + \mathfrak{S}_{abc} \left\{ q_{e_1 e_2}^u g_v^{e_1} g_w^{e_2} q_{ab}^v q_{cd}^w \right\} - \mathcal{S}_{ab,cd} \left\{ (q_{ab}^u g_v^0 + r_{abe}^u g_v^e) q_{cd}^v \right\}, \end{aligned}$$

with $\mathcal{S}_{a,b,cd} = \mathfrak{S}_{abcd} \cdot \mathfrak{S}_{bcd}$. This expression corrects typos in the formula for \tilde{r}_{abcd}^u in [7, p. 108]. As in the case of F_3 , the subspace $|F_4| \subset S^4 T_x^* X$ is well-defined modulo the ideal generated by $|II|, |F_3|$.

In this paper, we will be interested in the *special case* that $g_b^0 = 0 = g_b^a$; from (3.3, refeqn:tilde3) we see that these are the fibre motions that preserve the coefficients of II and F_3 . Then the fibre variation for the coefficients of F_4 is given by

$$(3.5) \quad \tilde{r}_{abcd}^u = r_{abcd}^u - \mathcal{S}_{ab,cd} (q_{ab}^u g_v^0 q_{cd}^v) .$$

4. THE TRACE-FREE, RANK 1 MATRICES

Let W be a complex vector space of dimension $m+1$, and let X denote the variety of trace-free, rank one linear maps $W \rightarrow W$. Observe that the rank one transformations may be identified with the Segre variety $\text{Seg}(\mathbb{P}W \times \mathbb{P}W^*) = \{[v \otimes w^*] \mid v \in W, w^* \in W^*\}$. The matrix $v \otimes w^*$ is trace-free if and only if $w^*(v) = 0$. That is, X is a hyperplane section of the Segre variety.

This variety is the unique closed orbit of the adjoint action on $\mathfrak{sl}(W)$. Let $\{v_j\}_{j=0}^m$ be a basis of W and $\{v_j^*\}$ the dual basis. Let $f = (f_0, \dots, f_m) \in \text{SL}(W)$, and write $f^{-1} = (f_0^*, \dots, f_m^*)^t$. Then the orbit of $v_0 \otimes v_m^* \in \mathfrak{sl}(W)$ under the adjoint action is

$$\{f_0 \otimes f_m^* \mid f \in \text{SL}(W)\} \subset \mathfrak{sl}(W);$$

and this is precisely the set of trace-free, rank one matrices.

Fix the index ranges

$$\begin{aligned} 0 &\leq j, k \leq m, \\ 1 &\leq \alpha, \beta \leq m-1, \\ m &\leq \bar{\alpha}, \bar{\beta} \leq 2m-2, \quad \bar{\alpha} = \alpha + (m-1), \\ 1 &\leq a, b \leq 2m-1 = n. \end{aligned}$$

The vectors

$$\begin{aligned} (4.1) \quad e_0 &= f_0 \otimes f_m^*; \\ e_\alpha &= f_\alpha \otimes f_m^*, \quad e_{\bar{\alpha}} = f_0 \otimes f_\alpha^*, \quad e_{2m-1} = \frac{1}{2} (f_m \otimes f_m^* - f_0 \otimes f_0^*); \\ e_{\alpha 0} &= f_\alpha \otimes f_0^*, \quad e_{m\alpha} = f_m \otimes f_\alpha^*, \quad e_{m0} = f_m \otimes f_0^*, \\ e_{\alpha\beta} &= f_\alpha \otimes f_\beta^* - \frac{1}{2} \delta_{\alpha\beta} (f_0 \otimes f_0^* + f_m \otimes f_m^*) \end{aligned}$$

span $\mathfrak{sl}(W)$, yielding a map $\varphi : \text{SL}(W) \rightarrow \text{GL}(\mathfrak{sl}(W))$.

Let $\mathcal{F} \subset \text{GL}(\mathfrak{sl}_{m+1})$ denote the set of all such framings. Observe that \mathcal{F} is a sub-bundle of the first-order adapted framings over X . To see this, let η be the Maurer-Cartan form on $\text{SL}(W)$, so that

$$df_j = \eta_j^k f_k, \quad \text{and} \quad df_j^* = -\eta_k^j f_k^*.$$

These 1-forms satisfy $\eta_j^j = 0$, and are otherwise linearly independent. Notice that $e_0 \in \widehat{X}$, and

$$de_0 = \eta_0^j f_j \otimes f_m^* - f_0 \otimes \eta_j^m f_j^* = (\eta_0^0 - \eta_m^m) e_0 + \eta_0^\alpha e_\alpha - \eta_\alpha^m e_{\bar{\alpha}} + 2 \eta_0^m e_{2m-1}.$$

This implies that the $e_\alpha, e_{\bar{\alpha}}, e_{2m-1}$ span $T_{e_0}\widehat{X}$, and we have a first-order adapted framing of $\mathfrak{sl}(W)$ over X .

4.1. The Maurer-Cartan form on \mathcal{F} . Now let ω denote the pull-back of the Maurer-Cartan form on $\mathrm{GL}(\mathfrak{sl}_{m+1})$ to \mathcal{F} , so that $de = \omega e$. Computations analogous to that of de_0 above assure us that the 1-forms $\omega_0^0, \omega_0^\alpha, \omega_\alpha^0$ and ω_β^α are linearly independent, and yield the following relations

(4.2)

$$\begin{aligned} \omega_\alpha^{2m-1} &= -\omega_0^{\bar{\alpha}}, \quad \omega_\alpha^{\beta 0} = -\frac{1}{2} \delta_\alpha^\beta \omega_0^{2m-1}, \quad \omega_\alpha^{\beta\gamma} = \delta_\alpha^\beta \omega_0^{\bar{\gamma}} \\ \omega_\alpha^{2m-1} &= \omega_0^\alpha, \quad \omega_\alpha^{m\beta} = \frac{1}{2} \delta_\alpha^\beta \omega_0^{2m-1}, \quad \omega_\alpha^{\beta\gamma} = \delta_\alpha^\gamma \omega_0^\beta, \quad \omega_\alpha^{\bar{\beta}} = -\omega_\beta^\alpha + \delta_{\alpha\beta} \omega_0^0 \\ \omega_{2m-1}^\alpha &= -\frac{1}{2} \omega_0^{\bar{\alpha}}, \quad \omega_{2m-1}^{\bar{\alpha}} = \frac{1}{2} \omega_0^\alpha, \quad \omega_{2m-1}^{\alpha 0} = -\frac{1}{2} \omega_0^\alpha, \quad \omega_{2m-1}^{m\alpha} = \frac{1}{2} \omega_0^{\bar{\alpha}}, \quad \omega_{2m-1}^{m0} = -\frac{1}{2} \omega_0^{2m-1} \\ \omega_{\alpha 0}^\beta &= -\delta_\alpha^\beta \omega_{2m-1}^0, \quad \omega_{\alpha 0}^{2m-1} = -\omega_\alpha^0, \quad \omega_{\alpha 0}^{\beta 0} = \omega_\alpha^\beta - \delta_\alpha^\beta \omega_0^0, \quad \omega_{\alpha 0}^{\beta\gamma} = -\delta_\alpha^\beta \omega_\gamma^0, \quad \omega_{\alpha 0}^{m0} = -\omega_0^{\bar{\alpha}} \\ \omega_{m\alpha}^{\bar{\beta}} &= \delta_\alpha^\beta \omega_{2m-1}^0, \quad \omega_{m\alpha}^{2m-1} = \omega_\alpha^0, \quad \omega_{m\alpha}^{m\beta} = -\omega_\beta^\alpha, \quad \omega_{m\alpha}^{\beta\gamma} = -\delta_\alpha^\gamma \omega_\beta^0, \quad \omega_{m\alpha}^{m0} = -\omega_\alpha^0 \\ \omega_{\alpha\beta}^\gamma &= \left(\delta_\alpha^\gamma \delta_\beta^\varepsilon + \frac{1}{2} \delta_{\alpha\beta} \delta^{\gamma\varepsilon} \right) \omega_\varepsilon^0, \quad \omega_{\alpha\beta}^{\bar{\gamma}} = \left(\delta_\alpha^\varepsilon \delta_\beta^{\bar{\gamma}} + \frac{1}{2} \delta_{\alpha\beta} \delta^{\gamma\varepsilon} \right) \omega_\varepsilon^0 \\ \omega_{\alpha\beta}^{\gamma 0} &= -\left(\delta_\alpha^\gamma \delta_\beta^\varepsilon + \frac{1}{2} \delta_{\alpha\beta} \delta_\varepsilon^\gamma \right) \omega_0^\varepsilon, \quad \omega_{\alpha\beta}^{m\gamma} = -\left(\delta_{\alpha\varepsilon} \delta_\beta^\gamma + \frac{1}{2} \delta_{\alpha\beta} \delta_\varepsilon^\gamma \right) \omega_0^{\bar{\varepsilon}}, \quad \omega_{\alpha\beta}^{\gamma\varepsilon} = \delta_\beta^\varepsilon \omega_\alpha^\gamma - \delta_\alpha^\gamma \omega_\beta^\varepsilon \\ \omega_{m0}^{2m-1} &= -2 \omega_{2m-1}^0, \quad \omega_{m0}^{\alpha 0} = -\omega_\alpha^0, \quad \omega_{m0}^{m\alpha} = -\omega_\alpha^0, \quad \omega_{m0}^{m0} = -\omega_0^0 \end{aligned}$$

The remaining 1-forms (those not appearing above) vanish on the pull-back.

These are precisely the equations of the φ -pullback of the Maurer-Cartan form on $\mathrm{GL}(\mathfrak{sl}(W))$. In fact, if $\eta = \eta_k^j E_j^k$ denotes the Maurer-Cartan form on $\mathrm{SL}(W)$, then

$$\begin{aligned} \eta_0^0 &= \frac{1}{2} \left(\frac{1}{n+1} \sum (\omega_\alpha^{\bar{\alpha}} - \omega_\alpha^\alpha) + \omega_0^0 \right) & \eta_\beta^0 &= \omega_\beta^0 & \eta_n^0 &= \omega_{2m-1}^0 \\ \eta_0^\alpha &= \omega_0^\alpha & \eta_\beta^\alpha &= \frac{1}{2} \left(\omega_\beta^\alpha - \omega_\alpha^{\bar{\beta}} + \frac{\delta_{\alpha\beta}}{n+1} \sum (\omega_\alpha^{\bar{\alpha}} - \omega_\alpha^\alpha) \right) & \eta_n^\alpha &= -\omega_\alpha^0 \\ \eta_0^n &= \frac{1}{2} \omega_0^{2m-1} & \eta_\beta^n &= -\omega_\beta^{\bar{\beta}} & \eta_n^n &= \frac{1}{2} \left(\frac{1}{n+1} \sum (\omega_\alpha^{\bar{\alpha}} - \omega_\alpha^\alpha) - \omega_0^0 \right). \end{aligned}$$

4.2. The differential invariants F_k . Recollect (1.2) that the coefficients q_{ab}^u of the second fundamental form are defined by $\omega_a^u = q_{ab}^u \omega_0^b$. Inspecting (4.2), we see that the non-zero coefficients are

$$(4.3) \quad q_{2m-1, 2m-1}^{m0} = -\frac{1}{2}, \quad q_{\beta, 2m-1}^{\alpha 0} = -\frac{1}{2} \delta_\beta^\alpha, \quad q_{\beta, 2m-1}^{m\alpha} = \frac{1}{2} \delta_\beta^\alpha, \quad q_{\gamma\bar{\varepsilon}}^{\alpha\beta} = \delta_\gamma^\alpha \delta_\varepsilon^\beta.$$

In particular,

$$|II| = \langle \omega_0^{2m-1} \omega_0^{2m-1}, \omega_0^\alpha \omega_0^{2m-1}, \omega_0^{\bar{\alpha}} \omega_0^{2m-1}, \omega_0^\alpha \omega_0^{\bar{\alpha}} \rangle \subset S^2 T_x X.$$

The first quadric in $|II|$ above implies that the cone over $\mathcal{C}_{2,x} = \mathrm{Baseloc}|II|$ lies in the *contact hyperplane* $T_1 := \langle \omega_0^{2m-1} \rangle^\perp \subset T_x X$. The fourth quadric tells us that $\mathcal{C}_{2,x}$, the set of lines osculating to order two at $x \in X$, is the disjoint union of two linear spaces

$$\mathcal{C}_{2,x} = \mathbb{P}^{m-1} \sqcup \mathbb{P}^{m-1} \subset \langle \omega_0^{2m-1} \rangle^\perp \subset T_x X.$$

It is straightforward to confirm that the two disjoint \mathbb{P}^{m-1} 's making up $\mathcal{C}_{2,x}$ correspond to integrable distributions, $D_1 = \{0 = \omega_0^\alpha\}$ and $D_2 = \{0 = \omega_0^{\bar{\alpha}}\}$, in T_1 .

Computations with (1.3, 1.5) show that the non-zero coefficients of F_3 are

$$(4.4) \quad r_{\beta\gamma\bar{\varepsilon}}^{\alpha 0} = \frac{1}{2} (\delta_\beta^\alpha \delta_{\gamma\varepsilon} + \delta_\gamma^\alpha \delta_{\beta\varepsilon}) \quad \text{and} \quad r_{\beta\gamma\bar{\varepsilon}}^{m\alpha} = \frac{1}{2} (\delta_\gamma^\alpha \delta_{\beta\varepsilon} + \delta_\varepsilon^\alpha \delta_{\beta\gamma});$$

and the nonzero coefficients of F_4 are

$$(4.5) \quad r_{\alpha\beta\gamma\epsilon}^{m0} = \frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\epsilon} + \delta_{\alpha\epsilon}\delta_{\beta\gamma}) = r_{\alpha\beta\gamma}^{\epsilon 0} = r_{\alpha\beta\gamma}^{m\epsilon}.$$

Therefore, $|F_{3,x}| = \langle \omega_0^\alpha (\sum \omega_0^\epsilon \omega_0^{\bar{\epsilon}}), \omega_0^{\bar{\alpha}} (\sum \omega_0^\epsilon \omega_0^{\bar{\epsilon}}) \rangle$ and $|F_{4,x}| = \langle (\sum \omega_0^\epsilon \omega_0^{\bar{\epsilon}})^2 \rangle$. Whence $\mathcal{C}_{4,x} = \mathcal{C}_{3,x} = \mathcal{C}_{2,x}$. Notice that cubics of $|F_{3,x}|$ are the derivatives of the $F_{4,x}$ quartic.

Finally, $F_k = 0$ for all $k \geq 5$. (Compute $F_k = 0$ directly for $5 \leq k \leq 9$, and then apply Corollary 1.6.) Hence $\mathcal{C}_x = \mathcal{C}_{2,x}$. That is, any line osculating to order two at $x \in X$ is necessarily contained in X . This is consistent with the fact that X is generated by degree two polynomials: the rank one matrices are given by the vanishing of their 2-by-2 minors.

The trace-free matrices are rigid to order three:

Theorem 4.6. *Let $Y^{2m-1} \subset \mathbb{P}(\mathfrak{sl}_{m+1}) = \mathbb{P}^{m^2+2m-1}$ be an algebraic variety, with $m > 1$. Suppose that there exists a framing e_y over a 3-general point $y \in Y$ at which the nonzero coefficients of II_Y and $F_{3,Y}$ are given by equations (4.3) and (4.4), respectively. Then Y is projectively equivalent to the variety of trace-free, rank one $(m+1) \times (m+1)$ matrices.*

Remark. This result is better than had been expected. Notice that the coefficients of F_4 can not be normalized to zero. This non-vanishing led Landsberg and Manivel to conjecture that the adjoint variety of $\mathrm{SL}_{m+1}\mathbb{C}$ was rigid to order four, but not to order three [12]. Indeed, this is the first example of a variety that is rigid to order k for which the higher order F_ℓ , $\ell > k$, can not all be normalized to zero.

Remark. It suffices to assume that the coefficients of II and F_3 may be put in the form (4.3,4.4) at a frame $e \in \mathcal{F}_Y^1$ over a general point $y \in Y$. (More precisely, y is a 3-general point: the discrete invariants associated to second and third order data should be constant in a neighborhood of y .)

Remark. Third order rigidity does not hold when $m = 1$. In this case we have

$$f = \begin{pmatrix} f_0^0 & f_1^0 \\ f_0^1 & f_1^1 \end{pmatrix} \quad \text{and} \quad f^{-1} = \begin{pmatrix} f_1^1 & -f_1^0 \\ -f_0^1 & f_0^0 \end{pmatrix};$$

and the orbit of $v_0 \otimes v_1^*$ under the adjoint action is

$$\left\{ f_0 \otimes f_1^* = \begin{pmatrix} -f_0^0 f_0^1 & f_0^0 f_0^0 \\ -f_0^1 f_0^1 & f_0^1 f_0^0 \end{pmatrix} : f_0 = \begin{pmatrix} f_0^0 \\ f_0^1 \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\} \right\}.$$

Notice that $f_0 \otimes f_1^*$ may be identified with the symmetric product $f_0 \circ f_0$. Therefore the adjoint variety of $\mathrm{SL}_2\mathbb{C}$ is the Veronese embedding $v_2(\mathbb{P}^1) \subset \mathbb{P}S^2\mathbb{C}^2 = \mathbb{P}^2$. The plane conics are rigid to order five (cf. §2.1.3).

5. THE PROOF OF THEOREM 4.6

The goal of this section is to show that the adjoint variety $X \subset \mathbb{P}(\mathfrak{sl}_{m+1})$ is rigid to order three. That is, if $Y^{2m-1} \subset \mathbb{P}^{m^2+2m-1}$ admits a sub-bundle \mathcal{F}_Y^3 over an open neighborhood $U \subset Y$ of y of the first-order adapted frame bundle on which $II_Y = II_X$ and $F_{3,Y} = F_{3,X}$, then Y is projectively equivalent to X . Our strategy is to reduce \mathcal{F}_Y^3 to a sub-bundle \mathcal{F}_Y^4 on which (i) the non-zero coefficients of $F_{4,Y}$ are given by (4.5), and (ii) the coefficients of $F_{k,Y}$ vanish for $5 \leq k \leq 9$. Then Corollary 1.6 and Proposition 2.1 yield the desired rigidity.

The proof is structured as follows. After deriving the consequences of third-order agreement,

- The fourth-order coefficients r_{abce}^{m0} are computed in §5.1. Then all coefficients (except $r_{\alpha\beta\gamma\bar{\varepsilon}}^{m0}$, which is given by (4.5)) are normalized to zero. We restrict to the sub-bundle $\mathcal{F}_Y^4 \subset \mathcal{F}_Y^3$ on which these normalizations hold.
- We show in §5.2 and §5.3 that the coefficients $r_{abce}^{\rho\zeta}$ and $r_{abce}^{\rho0}, r_{abce}^{m\zeta}$ vanish on \mathcal{F}_Y^4 , respectively.
- In §5.4 we see that the coefficients of $F_{k,Y}$, $k \geq 5$ are zero on \mathcal{F}_Y^4 .
- The calculations in §5.2–5.4 assume that $m > 2$. I address the case $m = 2$ in §5.5.

Let \mathcal{F}_U^3 be the sub-bundle of the first-order adapted frames on which the coefficients of II_Y and $F_{3,Y}$ are given by (4.3, 4.4). As before, let ω denote the pull-back of the Maurer-Cartan form on $\mathrm{GL}(\mathfrak{sl}_{m+1})$ to \mathcal{F}_U^3 . The condition $II_Y = II_X$ implies

$$(5.1) \quad \begin{aligned} \omega_\alpha^{\beta 0} &= -\frac{1}{2} \delta_\alpha^\beta \omega_0^{2m-1} & \omega_\alpha^{m\beta} &= 0 & \omega_\alpha^{\beta\gamma} &= \delta_\alpha^\beta \omega_0^{\bar{\gamma}} & \omega_\alpha^{m0} &= 0 \\ \omega_\alpha^{\beta 0} &= 0 & \omega_\alpha^{m\beta} &= \frac{1}{2} \delta_\alpha^\beta \omega_0^{2m-1} & \omega_\alpha^{\beta\gamma} &= \delta_\alpha^\gamma \omega_0^\beta & \omega_\alpha^{m0} &= 0 \\ \omega_{2m-1}^{\beta 0} &= -\frac{1}{2} \omega_0^\beta & \omega_{2m-1}^{m\beta} &= \frac{1}{2} \omega_0^{\bar{\beta}} & \omega_{2m-1}^{\beta\gamma} &= 0 & \omega_{2m-1}^{m0} &= -\frac{1}{2} \omega_0^{2m-1}. \end{aligned}$$

The condition $F_{3,Y} = F_{3,X}$ yields

$$(5.2) \quad \begin{aligned} 0 &= \omega_\beta^\alpha = \omega_{\bar{\beta}}^{\bar{\alpha}} = \omega_{m\beta}^{\alpha 0} = \omega_{\beta 0}^{m\alpha} = \omega_{m0}^{\alpha\beta} = \omega_{\alpha\beta}^{m0} \\ \omega_\alpha^{2m-1} &= \omega_{\alpha 0}^{m0} = -\omega_0^{\bar{\alpha}}, \quad \omega_\alpha^{2m-1} = -\omega_{m\alpha}^{m0} = \omega_0^\alpha, \quad \omega_0^0 + \omega_{m0}^{m0} = 2\omega_{2m-1}^{2m-1} \\ \omega_\beta^{\alpha 0} - \omega_\beta^\alpha &= \delta_\beta^\alpha (\omega_{2m-1}^{2m-1} - \omega_0^0), \quad \omega_{\beta\gamma}^{\alpha 0} = -\left(\delta_\beta^\alpha \delta_{\gamma\varepsilon} + \frac{1}{2} \delta_\varepsilon^\alpha \delta_{\beta\gamma}\right) \omega_0^\varepsilon, \quad \omega_{m0}^{\alpha 0} = 2\omega_{2m-1}^\alpha \\ \omega_{m\beta}^{\alpha 0} - \omega_{\bar{\beta}}^{\bar{\alpha}} &= \delta_\beta^\alpha (\omega_{2m-1}^{2m-1} - \omega_0^0), \quad \omega_{\beta\gamma}^{m\alpha} = -\left(\delta_\gamma^\alpha \delta_{\beta\varepsilon} + \frac{1}{2} \delta_\varepsilon^\alpha \delta_{\beta\gamma}\right) \omega_0^{\bar{\varepsilon}}, \quad \omega_{m0}^{m\alpha} = -2\omega_{2m-1}^{\bar{\alpha}} \\ \omega_{\gamma 0}^{\alpha\beta} &= -2\delta_\gamma^\alpha \omega_{2m-1}^{\bar{\beta}}, \quad \omega_{m\gamma}^{\alpha\beta} = 2\delta_\gamma^\beta \omega_{2m-1}^\alpha, \quad \omega_{\gamma\varepsilon}^{\alpha\beta} + \delta_\gamma^\alpha \delta_\varepsilon^\beta \omega_0^0 = \delta_\gamma^\alpha \omega_{\bar{\varepsilon}}^{\bar{\beta}} + \delta_\varepsilon^\beta \omega_\gamma^\alpha. \end{aligned}$$

I will use these relations without mention when computing the coefficients of $F_{4,Y}$ below.

5.1. $F_{4,Y}$ – **the conormal direction** $u = m0$. Direct computations with (1.5) yield

$$\begin{aligned} 0 &= r_{\alpha\beta\gamma e}^{m0} \omega_0^e = r_{\bar{\alpha}\bar{\beta}\bar{\gamma}e}^{m0} \omega_0^e = r_{\alpha\beta(2m-1)e}^{m0} \omega_0^e = r_{\bar{\alpha}\bar{\beta}(2m-1)e}^{m0} \omega_0^e \\ r_{\alpha\beta\bar{\gamma}e}^{m0} \omega_0^e &= \frac{1}{2} (\delta_{\alpha\gamma} \delta_{\beta\varepsilon} + \delta_{\alpha\varepsilon} \delta_{\beta\gamma}) \omega_0^{\bar{\varepsilon}} \\ r_{\alpha\bar{\beta}\bar{\gamma}e}^{m0} \omega_0^e &= \frac{1}{2} (\delta_{\alpha\beta} \delta_{\gamma\varepsilon} + \delta_{\alpha\gamma} \delta_{\beta\varepsilon}) \omega_0^\varepsilon \\ r_{\alpha\bar{\beta}(2m-1)e}^{m0} \omega_0^e &= \frac{1}{2} \omega_{\alpha\beta}^{2m-1} \\ r_{\alpha(2m-1)^2e}^{m0} \omega_0^e &= -\frac{1}{2} (\omega_{\alpha 0}^{2m-1} + \omega_\alpha^0) \\ r_{\bar{\alpha}(2m-1)^2e}^{m0} \omega_0^e &= \frac{1}{2} (\omega_{m\alpha}^{2m-1} - \omega_\alpha^0) \\ r_{(2m-1)^3e}^{m0} \omega_0^e &= -\frac{3}{2} (\omega_{2m-1}^0 + \frac{1}{2} \omega_{m0}^{2m-1}). \end{aligned}$$

Equation (3.5) permits us to normalize the coefficients of the last four equations above to zero through transformations of the form (3.1) with $g_a = 0 = g_u^a$ and

$$g_{\alpha 0}^0 = \frac{4}{3} r_{(2m-1)^3\alpha}^{m0}, \quad g_{m\alpha}^0 = -\frac{4}{3} r_{(2m-1)^3\bar{\alpha}}^{m0}, \quad g_{\alpha\beta}^0 = -2r_{\alpha\bar{\beta}(2m-1)^2}^{m0}, \quad g_{m0}^0 = \frac{2}{3} r_{(2m-1)^4}^{m0}.$$

(Note that equations (3.3, 3.4) assure us that the coefficients of II_Y and $F_{3,Y}$ are preserved.) Under these normalizations

$$(5.3) \quad \omega_{\alpha 0}^{2m-1} = -\omega_\alpha^0, \quad \omega_{m\alpha}^{2m-1} = \omega_\alpha^0, \quad \omega_{\alpha\beta}^{2m-1} = 0, \quad \omega_{m0}^{2m-1} = -2\omega_{2m-1}^0;$$

and the only non-zero coefficients of $F_{4,Y}$ in the conormal direction $u = m0$ are given by (4.5)

$$r_{\alpha\beta\bar{\gamma}\bar{\varepsilon}}^{m0} = \frac{1}{2} (\delta_{\alpha\gamma} \delta_{\beta\varepsilon} + \delta_{\alpha\varepsilon} \delta_{\beta\gamma}).$$

Restrict, from this point on, to the sub-bundle $\mathcal{F}_U^4 \subset \mathcal{F}_U^3$ on which these normalizations hold.

5.2. $F_{4,Y}$ – **the conormal direction** $u = \rho\zeta$. After simplifying with (5.2, 5.3), the coefficients $r_{abce}^{\rho\zeta}$ are given by (1.5) as

$$(5.4) \quad 0 = r_{(2m-1)^3e}^{\rho\zeta} \omega_0^e = r_{\alpha\beta\gamma e}^{\rho\zeta} \omega_0^e = r_{\bar{\alpha}\bar{\beta}\bar{\gamma}e}^{\rho\zeta} \omega_0^e$$

$$(5.5) \quad r_{\alpha(2m-1)^2e}^{\rho\zeta} \omega_0^e = \frac{1}{2} \delta_\alpha^\rho \omega_{m0}^\zeta$$

$$(5.6) \quad r_{\bar{\alpha}(2m-1)^2e}^{\rho\zeta} \omega_0^e = \frac{1}{2} \delta_{\bar{\alpha}}^\rho \omega_{m0}^\zeta$$

$$(5.7) \quad r_{\alpha\beta(2m-1)e}^{\rho\zeta} \omega_0^e = \frac{1}{2} \left(\delta_\alpha^\rho \omega_{\beta 0}^\zeta + \delta_\beta^\rho \omega_{\alpha 0}^\zeta \right)$$

$$(5.8) \quad r_{\alpha\bar{\beta}(2m-1)e}^{\rho\zeta} \omega_0^e = \delta_\alpha^\rho \delta_{\bar{\beta}}^\zeta \omega_{2m-1}^0 + \frac{1}{2} \delta_\beta^\zeta \omega_{\alpha 0}^\rho - \frac{1}{2} \delta_\alpha^\rho \omega_{m\beta}^\zeta$$

$$(5.9) \quad r_{\bar{\alpha}\bar{\beta}(2m-1)e}^{\rho\zeta} \omega_0^e = -\frac{1}{2} \left(\delta_\alpha^\zeta \omega_{m\beta}^\rho + \delta_\beta^\zeta \omega_{m\alpha}^\rho \right)$$

$$(5.10) \quad r_{\alpha\beta\bar{\gamma}e}^{\rho\zeta} \omega_0^e = \delta_\alpha^\rho \left(\delta_\gamma^\zeta \omega_\beta^0 + \delta_{\beta\gamma} \omega_{2m-1}^\zeta - \omega_{\beta\gamma}^\zeta \right) + \delta_\beta^\rho \left(\delta_\gamma^\zeta \omega_\alpha^0 + \delta_{\alpha\gamma} \omega_{2m-1}^\zeta - \omega_{\alpha\gamma}^\zeta \right)$$

$$(5.11) \quad r_{\alpha\bar{\beta}\bar{\gamma}e}^{\rho\zeta} \omega_0^e = \delta_\beta^\zeta \left(\delta_\alpha^\rho \omega_{\bar{\gamma}}^0 - \delta_{\alpha\gamma} \omega_{2m-1}^\rho - \omega_{\alpha\gamma}^\rho \right) + \delta_\gamma^\zeta \left(\delta_\alpha^\rho \omega_{\bar{\beta}}^0 - \delta_{\alpha\beta} \omega_{2m-1}^\rho - \omega_{\alpha\beta}^\rho \right).$$

The first three equations (5.4, 5.5, 5.6) force $0 = \omega_{m0}^\zeta = \omega_{m0}^\rho$. This is seen as follows. Equations (5.5, 5.6) imply that there exist functions r_e^ρ, r_e^ζ such that

$$(5.12) \quad r_{\alpha(2m-1)^2e}^{\rho\zeta} = \delta_\alpha^\rho r_e^\zeta \quad \text{and} \quad r_{\bar{\alpha}(2m-1)^2e}^{\rho\zeta} = \delta_{\bar{\alpha}}^\rho r_e^\zeta.$$

Immediately, the vanishing $0 = r_{(2m-1)^3e}^{\rho\zeta}$ of (5.4) yields $0 = r_{2m-1}^\rho = r_{2m-1}^\zeta$. Next, the symmetries $r_{\alpha(2m-1)^2\beta}^{\rho\zeta} = r_{\beta(2m-1)^2\alpha}^{\rho\zeta}$ and $r_{\bar{\alpha}(2m-1)^2\bar{\beta}}^{\rho\zeta} = r_{\bar{\beta}(2m-1)^2\bar{\alpha}}^{\rho\zeta}$ imply $\delta_\alpha^\rho r_\beta^\zeta = \delta_\beta^\rho r_\alpha^\zeta$ and $\delta_\alpha^\zeta r_\beta^\rho = \delta_\beta^\zeta r_\alpha^\rho$. If $m-1 \geq 2$, we may pick $\rho = \alpha \neq \beta$ to see that $0 = r_\beta^\zeta$. Similarly, $0 = r_\alpha^\rho$. Likewise, working with the symmetry $r_{\alpha(2m-1)^2\bar{\beta}}^{\rho\zeta} = r_{\bar{\beta}(2m-1)^2\alpha}^{\rho\zeta}$, we may deduce $0 = r_\alpha^\rho = r_\beta^\zeta$.

At this point we have shown that the functions r_e^ζ and r_e^ρ are identically zero. It follows from (5.5, 5.6, 5.12) that

$$(5.13) \quad r_{ab(2m-1)^2}^{\rho\zeta} = 0, \quad \text{and} \quad 0 = \omega_{m0}^\rho = \omega_{m0}^\zeta.$$

Remark. The case $m-1=1$ is addressed separately in §5.5.

5.3. $F_{4,Y}$ – **the conormal directions** $u = \rho 0, m\rho$. Equations (1.5, 5.2, 5.3, 5.13) yield

$$(5.14) \quad 0 = r_{(2m-1)^3e}^{\rho 0} \omega_0^e = r_{\bar{\alpha}\bar{\beta}ce}^{\rho 0} \omega_0^e = r_{\alpha\beta\gamma e}^{\rho 0} \omega_0^e$$

$$(5.15) \quad r_{\alpha(2m-1)^2e}^{\rho 0} \omega_0^e = -\frac{1}{2} \left(\omega_{\alpha 0}^\rho + \delta_\alpha^\rho \omega_{2m-1}^0 \right)$$

$$(5.16) \quad r_{\bar{\alpha}(2m-1)^2e}^{\rho 0} \omega_0^e = \frac{1}{2} \omega_{m\alpha}^\rho$$

$$(5.17) \quad r_{\alpha\beta(2m-1)e}^{\rho 0} \omega_0^e = \frac{1}{2} \left(\delta_\alpha^\rho \delta_{\beta\varepsilon} + \delta_\beta^\rho \delta_{\alpha\varepsilon} \right) \left(\omega_{2m-1}^\varepsilon - \frac{1}{2} \omega_\varepsilon^0 \right)$$

$$(5.18) \quad r_{\alpha\bar{\beta}(2m-1)e}^{\rho 0} \omega_0^e = \frac{1}{2} \left(\omega_{\alpha\bar{\beta}}^\rho + (\delta_\alpha^\rho \delta_{\beta\varepsilon} + \delta_\varepsilon^\rho \delta_{\alpha\beta}) \omega_{2m-1}^\varepsilon - \frac{1}{2} \delta_\alpha^\rho \omega_{\bar{\beta}}^0 \right)$$

$$(5.19) \quad r_{\alpha\beta\bar{\gamma}e}^{\rho 0} \omega_0^e = \frac{1}{2} \delta_\alpha^\rho \left(\omega_{\bar{\gamma}}^\beta + \omega_{\beta\bar{\gamma}}^\gamma - \delta_{\beta\gamma} \left(\omega_0^0 + \omega_{2m-1}^{2m-1} \right) \right) \\ + \frac{1}{2} \delta_\beta^\rho \left(\omega_{\bar{\gamma}}^\alpha + \omega_{\alpha\bar{\gamma}}^\gamma - \delta_{\alpha\gamma} \left(\omega_0^0 + \omega_{2m-1}^{2m-1} \right) \right),$$

and

$$(5.20) \quad 0 = r_{(2m-1)^3e}^{m\rho} \omega_0^e = r_{\alpha\beta ce}^{m\rho} \omega_0^e = r_{\alpha\beta\bar{\gamma}e}^{m\rho} \omega_0^e$$

$$(5.21) \quad r_{\alpha(2m-1)^2e}^{m\rho} \omega_0^e = \frac{1}{2} \omega_{\alpha 0}^{\bar{\rho}}$$

$$(5.22) \quad r_{\alpha(2m-1)^2e}^{m\rho} \omega_0^e = \frac{1}{2} (\delta_\alpha^\rho \omega_{2m-1}^0 - \omega_{m\alpha}^{\bar{\rho}})$$

$$(5.23) \quad r_{\alpha\beta(2m-1)e}^{m\rho} \omega_0^e = \frac{1}{2} \left(-\omega_{\alpha\beta}^{\bar{\rho}} + (\delta_\beta^\rho \delta_{\alpha\varepsilon} + \delta_\varepsilon^\rho \delta_{\alpha\beta}) \omega_{2m-1}^{\bar{\varepsilon}} + \frac{1}{2} \delta_\beta^\rho \omega_\alpha^0 \right)$$

$$(5.24) \quad r_{\alpha\bar{\beta}(2m-1)e}^{m\rho} \omega_0^e = \frac{1}{2} (\delta_\alpha^\rho \delta_{\beta\varepsilon} + \delta_\beta^\rho \delta_{\alpha\varepsilon}) (\omega_{2m-1}^{\bar{\varepsilon}} + \frac{1}{2} \omega_{\bar{\varepsilon}}^0)$$

$$(5.25) \quad r_{\alpha\beta\bar{\gamma}e}^{m\rho} \omega_0^e = \frac{1}{2} \delta_\beta^\rho \left(\omega_{\alpha\bar{\gamma}}^{\bar{\rho}} + \omega_\alpha^\gamma - \delta_{\alpha\gamma} (\omega_0^0 + \omega_{2m-1}^{2m-1}) \right) \\ + \frac{1}{2} \delta_\gamma^\rho \left(\omega_{\beta\bar{\gamma}}^{\bar{\rho}} + \omega_\beta^\gamma - \delta_{\alpha\beta} (\omega_0^0 + \omega_{2m-1}^{2m-1}) \right).$$

Let's consider the various expressions for $\delta_\alpha^\rho \delta_\beta^\zeta \omega_{2m-1}^0 + \frac{1}{2} \delta_\beta^\zeta \omega_{\alpha 0}^\rho - \frac{1}{2} \delta_\alpha^\rho \omega_{m\beta}^{\bar{\zeta}}$ given above. From (5.8), and then (5.13):

$$\delta_\alpha^\rho \delta_\beta^\zeta \omega_{2m-1}^0 + \frac{1}{2} \delta_\beta^\zeta \omega_{\alpha 0}^\rho - \frac{1}{2} \delta_\alpha^\rho \omega_{m\beta}^{\bar{\zeta}} = r_{\alpha\bar{\beta}(2m-1)e}^{\rho\zeta} \omega_0^e = r_{\alpha\bar{\beta}(2m-1)\varepsilon}^{\rho\zeta} \omega_0^\varepsilon + r_{\alpha\bar{\beta}(2m-1)\bar{\varepsilon}}^{\rho\zeta} \omega_0^{\bar{\varepsilon}}.$$

And with (5.15, 5.22) and (5.14, 5.20):

$$\delta_\alpha^\rho \delta_\beta^\zeta \omega_{2m-1}^0 + \frac{1}{2} \delta_\beta^\zeta \omega_{\alpha 0}^\rho - \frac{1}{2} \delta_\alpha^\rho \omega_{m\beta}^{\bar{\zeta}} = \left(\delta_\alpha^\rho r_{\beta(2m-1)^2e}^{m\zeta} - \delta_\beta^\zeta r_{\alpha(2m-1)^2e}^{\rho 0} \right) \omega_0^e \\ = \left(\delta_\alpha^\rho r_{\beta(2m-1)^2\varepsilon}^{m\zeta} - \delta_\beta^\zeta r_{\alpha(2m-1)^2\varepsilon}^{\rho 0} \right) \omega_0^\varepsilon + \left(\delta_\alpha^\rho r_{\beta(2m-1)^2\bar{\varepsilon}}^{m\zeta} - \delta_\beta^\zeta r_{\alpha(2m-1)^2\bar{\varepsilon}}^{\rho 0} \right) \omega_0^{\bar{\varepsilon}}.$$

A comparison of these expressions yields

$$r_{\alpha\bar{\beta}(2m-1)\varepsilon}^{\rho\zeta} = \delta_\alpha^\rho r_{\beta(2m-1)^2\varepsilon}^{m\zeta} - \delta_\beta^\zeta r_{\alpha(2m-1)^2\varepsilon}^{\rho 0} \\ r_{\alpha\bar{\beta}(2m-1)\bar{\varepsilon}}^{\rho\zeta} = \delta_\alpha^\rho r_{\beta(2m-1)^2\bar{\varepsilon}}^{m\zeta} - \delta_\beta^\zeta r_{\alpha(2m-1)^2\bar{\varepsilon}}^{\rho 0}.$$

The symmetry in (α, ε) on the left side of the first equation, and the symmetry in $(\bar{\beta}, \bar{\varepsilon})$ on the left side of the second equation force

$$(5.26) \quad r_{\bar{\beta}(2m-1)^2\varepsilon}^{m\zeta} = 0 = r_{\alpha(2m-1)^2\bar{\varepsilon}}^{\rho 0},$$

respectively. This updates the formulas above to

$$(5.27) \quad \delta_\alpha^\rho \delta_\beta^\zeta \omega_{2m-1}^0 + \frac{1}{2} \delta_\beta^\zeta \omega_{\alpha 0}^\rho - \frac{1}{2} \delta_\alpha^\rho \omega_{m\beta}^{\bar{\zeta}} = r_{\alpha\bar{\beta}(2m-1)\varepsilon}^{\rho\zeta} \omega_0^\varepsilon + r_{\alpha\bar{\beta}(2m-1)\bar{\varepsilon}}^{\rho\zeta} \omega_0^{\bar{\varepsilon}} \\ = -\delta_\beta^\zeta r_{\alpha(2m-1)^2\varepsilon}^{\rho 0} \omega_0^\varepsilon + \delta_\alpha^\rho r_{\bar{\beta}(2m-1)^2\bar{\varepsilon}}^{m\zeta} \omega_0^{\bar{\varepsilon}}.$$

At this point we have

$$(5.28) \quad \frac{1}{2} \omega_{m\alpha}^\rho = r_{\alpha(2m-1)^2e}^{\rho 0} \omega_0^e = 0.$$

The first equality is just (5.16). The second equality is a consequence of (5.14, 5.26). Similarly, (5.20, 5.21, 5.26) yield

$$(5.29) \quad \frac{1}{2} \omega_{\alpha 0}^{\bar{\rho}} = r_{\alpha(2m-1)^2e}^{m\rho} \omega_0^e = 0.$$

These two equations, in conjunction with (5.7, 5.9), yield

$$r_{\alpha\bar{\beta}(2m-1)e}^{\rho\zeta} = 0 = r_{\alpha\beta(2m-1)e}^{\rho\zeta}.$$

This, and (5.13), implies

$$r_{\alpha\bar{\beta}(2m-1)e}^{\rho\zeta} = 0.$$

Now, from (5.27), we may conclude

$$r_{\alpha(2m-1)^2\varepsilon}^{\rho 0} = 0 = r_{\beta(2m-1)^2\bar{\varepsilon}}^{m\zeta}.$$

The first equality and (5.14, 5.15, 5.26) give us

$$(5.30) \quad \omega_{\alpha 0}^\rho + \delta_\alpha^\rho \omega_{2m-1}^0 = 0.$$

Similarly, the second equality and (5.20, 5.22, 5.26) yield

$$(5.31) \quad \omega_{m\alpha}^{\bar{\rho}} - \delta_a^\rho \omega_{2m-1}^0 = 0.$$

Finally, with (5.8, 5.30, 5.31) we have $r_{\alpha\beta(2m-1)e}^{\rho\zeta} = 0$.

Let's pause for a moment to assess our progress toward showing that the coefficients r_{abce}^u vanish ($u = \rho\zeta, \rho 0, n\rho$).

- The vanishing of the coefficients in (5.5–5.9) is now ensured by (5.13, 5.28–5.31). Notice in particular that the only potentially non-zero coefficients corresponding to the conormal direction $u = \rho\zeta$ are $r_{\alpha\beta\bar{\gamma}\bar{\varepsilon}}^{\rho\zeta}$.
- The vanishing of the coefficients in (5.15, 5.16) is given by (5.28, 5.30).
- The vanishing of the coefficients in (5.21, 5.22) is equivalent to (5.29, 5.31).

It remains to address the coefficients appearing in (5.10, 5.11, 5.17–5.19, 5.23–5.25).

The next portion of the analysis focuses on the equations (5.19, 5.25). They both (individually) imply that there are functions $r_{\beta\bar{\gamma}e}$ such that

$$\omega_{\bar{\gamma}}^{\bar{\beta}} + \omega_{\beta}^{\gamma} - \delta_{\beta\gamma} (\omega_0^0 + \omega_{2m-1}^{2m-1}) = r_{\beta\bar{\gamma}e} \omega_0^e.$$

In particular,

$$r_{\alpha\beta\bar{\gamma}e}^{\rho 0} = \frac{1}{2} \delta_\alpha^\rho r_{\beta\bar{\gamma}e} + \frac{1}{2} \delta_\beta^\rho r_{\alpha\bar{\gamma}e} \quad \text{and} \quad r_{\alpha\beta\bar{\gamma}e}^{m\rho} = \frac{1}{2} \delta_\beta^\rho r_{\alpha\bar{\gamma}e} + \frac{1}{2} \delta_\gamma^\rho r_{\alpha\bar{\beta}e}.$$

The symmetry of $r_{\alpha\beta\bar{\gamma}\bar{\varepsilon}}^{\rho 0}$ in $(\bar{\gamma}, \bar{\varepsilon})$, and the symmetry of $r_{\alpha\beta\bar{\gamma}\varepsilon}^{m\rho}$ in (α, ε) imply $r_{\beta\bar{\gamma}\bar{\varepsilon}}$ is symmetric in $(\bar{\gamma}, \bar{\varepsilon})$, and $r_{\alpha\bar{\gamma}\varepsilon}$ is symmetric in (α, ε) , respectively.

Now the symmetry of $r_{\alpha\beta\bar{\gamma}\varepsilon}^{\rho 0}$ in (β, ε) yields $r_{\alpha\bar{\gamma}\varepsilon} = 0$; and the symmetry of $r_{\alpha\beta\bar{\gamma}\bar{\varepsilon}}^{m\rho}$ in $(\bar{\gamma}, \bar{\varepsilon})$ yields $r_{\alpha\bar{\beta}\bar{\gamma}} = 0$. It is now a consequence of the equations (5.14, 5.19, 5.20, 5.25) that

$$r_{abce}^{\rho 0}, r_{abce}^{m\rho} = 0 \quad \text{if } a, b, c, e \neq 2m-1.$$

In fact, the only remaining, potentially non-zero, coefficients $r_{abce}^{\rho 0}, r_{abce}^{m\rho}$ are $r_{\alpha\beta\bar{\gamma}(2m-1)}^{\rho 0}$ and $r_{\alpha\beta\bar{\gamma}(2m-1)}^{m\rho}$.

This brings us to the final stage of our analysis of the coefficients of $F_{4,Y}$. From (5.17), we see that there are functions $r_{\alpha\bar{\varepsilon}}$ so that

$$\omega_{2m-1}^{\bar{\alpha}} - \frac{1}{2} \omega_\alpha^0 = r_{\alpha\bar{\varepsilon}} \omega_0^{\bar{\varepsilon}}.$$

In particular, (5.23) implies

$$\left(r_{\alpha\bar{\beta}(2m-1)\bar{\varepsilon}}^{m\rho} - \frac{1}{2} \delta_\beta^\rho r_{\alpha\bar{\varepsilon}} \right) \omega_0^{\bar{\varepsilon}} = \frac{1}{2} \left(-\omega_{\alpha\beta}^{\bar{\rho}} + \delta_{\alpha\beta} \omega_{2m-1}^{\bar{\rho}} + \delta_\beta^\rho \omega_\alpha^0 \right);$$

which, with (5.10), allows us to write

$$r_{\alpha\beta\bar{\gamma}\bar{\varepsilon}}^{\rho\zeta} = 2 \delta_\alpha^\rho \left(r_{\beta\bar{\gamma}(2m-1)\bar{\varepsilon}}^{m\zeta} - \frac{1}{2} \delta_\gamma^\zeta r_{\beta\bar{\varepsilon}} \right) + 2 \delta_\beta^\rho \left(r_{\alpha\bar{\gamma}(2m-1)\bar{\varepsilon}}^{m\zeta} - \frac{1}{2} \delta_\gamma^\zeta r_{\alpha\bar{\varepsilon}} \right).$$

The symmetry in $(\bar{\gamma}, \bar{\varepsilon})$ on the right forces $r_{\alpha\bar{\varepsilon}} = 0$, and we may conclude

$$(5.32) \quad 0 = \omega_{2m-1}^{\bar{\alpha}} - \frac{1}{2} \omega_\alpha^0$$

$$(5.33) \quad \frac{1}{2} r_{\alpha\beta\bar{\gamma}\bar{\varepsilon}}^{\rho\zeta} = \delta_\alpha^\rho r_{\beta\bar{\gamma}(2m-1)\bar{\varepsilon}}^{m\zeta} + \delta_\beta^\rho r_{\alpha\bar{\gamma}(2m-1)\bar{\varepsilon}}^{m\zeta}.$$

The analogous argument with (5.11, 5.18, 5.24) yields

$$(5.34) \quad 0 = \omega_{2m-1}^\alpha + \frac{1}{2} \omega_\alpha^0.$$

With (5.17, 5.32) we deduce that $r_{\alpha\beta(2m-1)e}^{\rho 0} = 0$, and we may now conclude that all the coefficients $r_{abce}^{\rho 0}$ vanish. In particular, (5.19) yields

$$(5.35) \quad \omega_\beta^\alpha + \omega_{\bar{\alpha}}^{\bar{\beta}} = \delta_{\alpha\beta} (\omega_0^0 + \omega_{2m-1}^{2m-1}).$$

Similarly, the vanishing of $r_{abce}^{m\rho}$ is a consequence of (5.24, 5.34). And this, along with (5.33), yields $r_{abce}^{\rho\zeta} = 0$. Therefore the only nonzero coefficients of $F_{4,Y}$ are given by (4.5) and we have established

$$F_{4,Y} = F_{4,X}.$$

Finally, note that $r_{abce}^{\rho\zeta} = 0$ and (5.10, 5.11, 5.32, 5.34) provide us with

$$(5.36) \quad \begin{aligned} \omega_{\alpha\beta}^\rho &= \delta_\alpha^\rho \omega_\beta^0 + \frac{1}{2} \delta_{\alpha\beta} \omega_\rho^0 \\ \omega_{\alpha\beta}^\zeta &= \delta_\beta^\zeta \omega_\alpha^0 + \frac{1}{2} \delta_{\alpha\beta} \omega_\zeta^0. \end{aligned}$$

5.4. Computations of $F_{k,Y}$, $k \geq 5$. In this section I will show that the higher order invariants $F_{k,Y}$, $k \geq 5$, vanish on \mathcal{F}_U^4 (the sub-bundle of \mathcal{F}_U^3 on which the normalizations of 5.1 hold). We begin with the coefficients of $F_{5,Y}$, which are given by Proposition 1.4. I will use the relations (5.2, 5.3, 5.13, 5.28–5.36) without mention.

Start with the coefficients corresponding to the conormal direction $u = \rho\zeta$; we will see that the ω_v^0 vanish. First,

$$\begin{aligned} 0 &= r_{(2m-1)^3ae}^{\rho\zeta} \omega_0^e = r_{(2m-1)^2\alpha\beta e}^{\rho\zeta} \omega_0^e = r_{(2m-1)^2\bar{\alpha}\bar{\beta}e}^{\rho\zeta} \omega_0^e \\ \frac{1}{2} \delta_\alpha^\rho \delta_\beta^\zeta \omega_{m0}^0 &= r_{(2m-1)^2\alpha\bar{\beta}e}^{\rho\zeta} \omega_0^e. \end{aligned}$$

Symmetry in the lower indices of r_{abcde}^u (and $m-1 > 1$) forces

$$\omega_{m0}^0 = 0.$$

Next,

$$\begin{aligned} r_{(2m-1)\alpha\beta\gamma e}^{\rho\zeta} \omega_0^e &= r_{(2m-1)\bar{\alpha}\bar{\beta}\bar{\gamma}e}^{\rho\zeta} \omega_0^e = 0 \\ r_{(2m-1)\alpha\beta\bar{\gamma}e}^{\rho\zeta} \omega_0^e &= \frac{1}{2} \delta_\gamma^\zeta (\delta_\alpha^\rho \omega_{\beta 0}^0 + \delta_\beta^\rho \omega_{\alpha 0}^0) \\ r_{(2m-1)\alpha\bar{\beta}\bar{\gamma}e}^{\rho\zeta} \omega_0^e &= -\frac{1}{2} \delta_\alpha^\rho (\delta_\beta^\zeta \omega_{m\gamma}^0 + \delta_\gamma^\zeta \omega_{m\beta}^0). \end{aligned}$$

As before, symmetry forces

$$0 = \omega_{\alpha 0}^0 = \omega_{m\beta}^0.$$

Finally,

$$\begin{aligned} 0 &= r_{\alpha\beta\gamma\vartheta e}^{\rho\zeta} \omega_0^e = r_{\alpha\beta\gamma\bar{\vartheta}e}^{\rho\zeta} \omega_0^e = r_{\alpha\bar{\beta}\bar{\gamma}\bar{\vartheta}e}^{\rho\zeta} \omega_0^e = r_{\bar{\alpha}\bar{\beta}\bar{\gamma}\vartheta e}^{\rho\zeta} \omega_0^e \\ r_{\alpha\bar{\beta}\bar{\gamma}\vartheta e}^{\rho\zeta} \omega_0^e &= -\delta_\alpha^\rho \delta_\gamma^\zeta \omega_{\beta\vartheta}^0 - \delta_\alpha^\rho \delta_\vartheta^\zeta \omega_{\beta\gamma}^0 - \delta_\beta^\rho \delta_\gamma^\zeta \omega_{\alpha\vartheta}^0 - \delta_\beta^\rho \delta_\vartheta^\zeta \omega_{\alpha\gamma}^0, \end{aligned}$$

and once again symmetry forces

$$\omega_{\alpha\beta}^0 = 0.$$

We conclude that the coefficients of $F_{5,Y}$ corresponding to the conormal direction $u = \rho\zeta$ vanish.

Straightforward, if lengthy, computations show that the remaining coefficients vanish as well. (*Remark.* Neither these computations, nor those that follow, require $m-1 > 1$.)

Additional calculations with Proposition 1.4 yield

$$F_{6,Y}, F_{7,Y}, F_{8,Y}, F_{9,Y} = 0,$$

completing the proof of Theorem 4.6 (in the case $m > 2$).

5.5. When $m = 2$. Since $m = 2$, we have $\alpha = 1$ and $\bar{\alpha} = \bar{1} = 2$. For consistency I will continue to use the notation $\alpha, \bar{\alpha}$, rather than $1, 2$, but will abbreviate $2m - 1 = 3$. As before $1 \leq c, e \leq 2m - 1 = 3$. To complete the proof of Theorem 4.6 we need to do two things:

- (1) show that $r_{abce}^u = 0$ for $u = \alpha 0, m\alpha, \alpha\alpha$, and
- (2) show that $\omega_u^0 = 0$ for $u = \alpha 0, m\alpha, \alpha\alpha, m0$.

First, (1.5) yields

$$\begin{aligned} 0 &= r_{\alpha\alpha e}^{\alpha 0} = r_{\bar{\alpha}\alpha e}^{\alpha 0} = r_{\alpha\alpha e}^{m\alpha} = r_{\bar{\alpha}\alpha e}^{m\alpha} = r_{\alpha\alpha e}^{\alpha\alpha} = r_{\bar{\alpha}\alpha e}^{\alpha\alpha} = r_{33e}^{\alpha\alpha}, \\ \omega_{m0}^\alpha &= 2r_{\bar{\alpha}33e}^{\alpha\alpha} \omega_0^e = -\frac{4}{3}r_{333e}^{\alpha 0} \omega_0^e \\ \omega_{m0}^{\bar{\alpha}} &= 2r_{\alpha 33e}^{\alpha\alpha} \omega_0^e = \frac{4}{3}r_{333e}^{m\alpha} \omega_0^e \\ \omega_{\alpha 0}^{\bar{\alpha}} &= r_{\alpha\alpha 3e}^{\alpha\alpha} \omega_0^e = 2r_{\alpha 33e}^{m\alpha} \omega_0^e \\ \omega_3^0 + \frac{1}{2}\omega_{\alpha 0}^\alpha - \frac{1}{2}\omega_{m\alpha}^{\bar{\alpha}} &= r_{\bar{\alpha}\alpha 3e}^{\alpha\alpha} \omega_0^e = r_{\bar{\alpha}33e}^{m\alpha} \omega_0^e - r_{\alpha 33e}^{\alpha 0} \omega_0^e \\ \omega_{m\alpha}^\alpha &= -r_{\bar{\alpha}\alpha 3e}^{\alpha\alpha} \omega_0^e = 2r_{\bar{\alpha}33e}^{\alpha 0} \omega_0^e \\ \omega_\alpha^0 + \omega_3^{\bar{\alpha}} - \omega_{\alpha\alpha}^{\bar{\alpha}} &= \frac{1}{2}r_{\alpha\alpha\bar{\alpha}e}^{\alpha\alpha} \omega_0^e = 2r_{\alpha\alpha 3e}^{m\alpha} \omega_0^e - r_{\alpha\alpha 3e}^{\alpha 0} \omega_0^e \\ \omega_\alpha^0 - \omega_3^\alpha - \omega_{\alpha\alpha}^\alpha &= \frac{1}{2}r_{\alpha\alpha\bar{\alpha}e}^{\alpha\alpha} \omega_0^e = -2r_{\alpha\alpha 3e}^{\alpha 0} \omega_0^e + r_{\alpha\alpha 3e}^{m\alpha} \omega_0^e \\ \omega_\alpha^\alpha + \omega_{\bar{\alpha}}^{\bar{\alpha}} - \omega_0^0 - \omega_3^3 &= r_{\alpha\alpha\bar{\alpha}e}^{m\alpha} \omega_0^e = r_{\alpha\alpha\bar{\alpha}e}^{\alpha 0} \omega_0^e. \end{aligned}$$

These relations force the coefficients to vanish, establishing (1).

Next calculations with Proposition 1.4 produce

$$0 = r_{333e}^{\alpha\alpha} \omega_0^e = r_{\alpha\alpha 33e}^{\alpha\alpha} \omega_0^e = r_{\bar{\alpha}\alpha 33e}^{\alpha\alpha} \omega_0^e, \quad \text{and} \quad r_{\alpha\bar{\alpha} 33e}^{\alpha\alpha} \omega_0^e = \frac{1}{4}\omega_{m0}^0.$$

Symmetry in the lower indices of $r_{abce}^{\alpha\alpha}$ forces

$$\omega_{m0}^0 = 0.$$

Additional computations yield

$$\begin{aligned} \omega_{\alpha 0}^0 &= -r_{\alpha\alpha 33\bar{\alpha}}^{\alpha 0} \omega_0^{\bar{\alpha}} = 2r_{\alpha\bar{\alpha} 33\bar{\alpha}}^{m\alpha} \omega_0^{\bar{\alpha}} = r_{\alpha\alpha\bar{\alpha} 3\bar{\alpha}}^{\alpha\alpha} \omega_0^{\bar{\alpha}} = -\frac{4}{3}r_{\alpha 333\bar{\alpha}}^{m0} \omega_0^{\bar{\alpha}} \\ \omega_{m\alpha}^0 &= 2r_{\alpha\bar{\alpha} 33\alpha}^{\alpha 0} \omega_0^\alpha = -r_{\bar{\alpha}\alpha 33\alpha}^{m\alpha} \omega_0^\alpha = -r_{\alpha\bar{\alpha}\alpha 3\alpha}^{\alpha\alpha} \omega_0^\alpha = \frac{4}{3}r_{\bar{\alpha} 333\alpha}^{m0} \omega_0^\alpha \\ \omega_{\alpha\alpha}^0 &= 2r_{\alpha\alpha\bar{\alpha} 33}^{\alpha 0} \omega_0^3 = -r_{\alpha\bar{\alpha}\alpha 33}^{m\alpha} \omega_0^3 = -\frac{1}{4}r_{\alpha\alpha\bar{\alpha}\alpha 3}^{\alpha\alpha} \omega_0^3 = 2r_{\alpha\bar{\alpha} 333}^{m0} \omega_0^3. \end{aligned}$$

And again the coefficients, and therefore the ω_u^0 , must vanish.

6. CONCLUDING REMARKS

The Veronese embedding $v_2(\mathbb{P}^{2m-1})$ of \mathbb{P}^{2m-1} may be identified with the adjoint variety of $\mathrm{Sp}_{2m}\mathbb{C}$, which is known to be rigid to order three (c.f. §2.1.2). So it is natural to ask if the adjoint varieties of the simple Lie groups are all rigid to order three.

There is some reason to hope that this is the case, as there are many similarities amongst these spaces: Given the adjoint variety of a simple Lie group it is the case that

- The F_k may be normalized to zero, $k \geq 5$.
- On the reduced frame bundle there is a single Fubini quartic $\mathbb{P}|F_4| \in S^4 T_{1,x}^*$. Here, as in §4.2, $T_{1,x} \subset T_x X$ is a contact hyperplane.
- The Fubini cubics $|F_3| \subset S^3 T_{1,x}$ are the derivatives of $|F_4|$.
- $\mathcal{C}_{2,x} = \mathcal{C}_x \subset \mathbb{P}T_{1,x}$.

The adjoint varieties of $\mathrm{SL}_{m+1}\mathbb{C}$ and $\mathrm{Sp}_{2m}\mathbb{C}$ are degenerate in the following sense. For $\mathrm{Sp}_{2m}\mathbb{C}$, that single Fubini quartic is zero, and $\mathcal{C}_{2,x} = \emptyset$ (cf. §2.1.2 and [10]). In the case of $\mathrm{SL}_{m+1}\mathbb{C}$, $\mathcal{C}_{2,x} = \mathbb{P}^{m-1} \sqcup \mathbb{P}^{m-1} \subset \mathbb{P}T_{1,x}$ is the disjoint union of two linear spaces, and the Fubini quartic factors as the square of two quadrics. (See §4.2.) The adjoint representation fails to be fundamental for these groups.

The adjoint representation is fundamental for the remaining simple Lie groups. As a consequence,

- $\mathcal{C}_{2,x}$ is a generalized minuscule variety, and the closed orbit of a semi-simple $H \subset G$ in $\mathbb{P}T_{1,x}$.
- The Fubini quartic is irreducible, and its zero locus in $\mathbb{P}T_{1,x}$ is the tangential variety of $\mathcal{C}_{2,x}$.

See [12] for details.

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